# PROBABILITY MEASURES AND MILYUTIN MAPS BETWEEN METRIC SPACES

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Abstract. We prove that the functor $\hat{P}$ of Radon probability measures transforms any open map between completely metrizable spaces into a soft map. This result is applied to establish some properties of Milyutin maps between completely metrizable spaces.

## 1. Introduction

In this paper we deal with metrizable spaces and continuous maps. By a (complete) space we mean a (completely) metrizable space, and by a measure a probability Radon measure. Recall that a measure $\mu$ on $X$ is said to be:

- *probability* if $\mu(X) = 1$;
- *Radon* if $\mu(B) = \sup\{\mu(K) : K \subset B \text{ and } K \text{ is compact}\}$ for any Borel set $B \subset X$;

The support *supp* $\mu$ of a measure $\mu$ is the intersection of all closed subsets $A$ of $X$ with $\mu(A) = \mu(X)$. It is well known that the support of any measure is non-empty and separable.

Everywhere below $\hat{P}(X)$ stands for the space of all probability Radon measures on $X$ equipped with the weak topology with respect to $C^*(X)$. Here, $C^*(X)$ is the space of bounded continuous functions on $X$ with the uniform convergence topology. According to [2], $\hat{P}$ is a functor in the category of metrizable spaces and continuous maps. In particular, for any map $f\colon X \to Y$ there exists a map $\hat{P}(f)\colon \hat{P}(X) \to \hat{P}(Y)$. A systematic study of the functor $\hat{P}$ can be found in [2] and [3]. We also consider the subspace $P_\beta(X) \subset \hat{P}(X)$ consisting of all measures $\mu$ such that *supp* $\mu$ is compact.

This paper is devoted to some properties of Milyutin maps between metrizable spaces. We say that $f\colon X \to Y$ is a *Milyutin map* if there

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exists a map $g\colon Y \to \hat{P}(X)$ such that *supp* $g(y) \subset f^{-1}(y)$ for every $y \in Y$. Such $g$ is called a choice map associated with $f$. According to [3, Theorem 3.15], for any metrizable $X$ there exists a barycentric map $b_{\hat{P}(X)}\colon \hat{P}(\hat{P}(X)) \to \hat{P}(X)$ such that $b_{\hat{P}(X)}(\nu) = \nu$ for all $\nu \in \hat{P}(X)$. Hence, if $g$ is a choice map associated with $f$, then the map $b_{\hat{P}(X)} \circ \hat{P}(g)\colon \hat{P}(Y) \to \hat{P}(X)$ is a right inverse of $\hat{P}(f)$. Consequently, $f$ is a Milyutin map if and only if $\hat{P}(f)$ admits a right inverse.

Our first principal result concerns the question when $\hat{P}(f)$ is a soft map. Recall that a map $f\colon X \to Y$ is soft if for any space $Z$ and its closed subset $A$ and any maps $g\colon Z \to Y$, $h\colon A \to X$ with $(f\circ h)|A = g$ there exists a map $\bar{g}\colon Z \to X$ such that $\bar{g}$ extends $h$ and $f \circ \bar{g} = g$. It is easily seen that every soft map is surjective and open.

**Theorem 1.1.** *Let $f\colon X \to Y$ be a surjective open map between complete spaces. Then $\hat{P}(f)\colon \hat{P}(X) \to \hat{P}(Y)$ is a soft map.*

The particular cases of Theorem 1.1 when both $X$ and $Y$ are either compact or separable were established in [8] and [4], respectively.

Since any soft map admits a right inverse, a map $f$ satisfying the hypotheses of Theorem 1.1 is a Milyutin map. We apply Theorem 1.1 to obtain some results about atomless and exact Milyutin maps introduced in [14]. If $f\colon X \to Y$ is a Milyutin map and there exists a choice map $g$ such that *supp* $g(y) = f^{-1}(y)$ (resp., $g(y)$ is an atomless measure on $f^{-1}(y)$ for each $y \in Y$, i.e. $g(y)(\{x\}) = 0$ for all $x \in f^{-1}(y)$), then $f$ is said to be an *exact* (resp., *atomless*) Milyutin map. It was established in [14] that, in the realm of Polish spaces $X$ and $Y$, $f$ is exact Milyutin if and only if it is open. The classes of atomless exact Milyutin maps and atomless Milyutin maps between Polish spaces were characterized in [1, Theorem 1.6]. The first class consists of all open maps possessing perfect fibers (i.e., without isolated points) [1, Theorem 1.6], and the second one of all maps $f\colon X \to Y$ such that for some Polish space $X_0 \subset X$ the restriction $f_0 = f|X_0\colon X_0 \to Y$ is an open surjection whose fibers are perfect [1, Theorem 1.7].

Next theorem is a non-separable analogue of [1, Theorem 1.7].

**Theorem 1.2.** *A continuous surjection $f\colon X \to Y$ of complete spaces is an atomless Milyutin map if and only if there exists a complete subspace $X_0 \subset X$ such that $f_0 = f|X_0\colon X_0 \to Y$ is an open surjection and all fibers of $f_0$ are perfect sets. Moreover, for any such $f$ there exists a map $f^*\colon P_\beta(Y) \to \hat{P}(X)$ such that any $f^*(\mu)$ is atomless and $\hat{P}(f)\big(f^*(\mu)\big) = \mu$, $\mu \in P_\beta(Y)$.*

We do not know whether under the hypotheses in Theorem 1.2 there exists a map $f^*\colon \hat{P}(Y) \to \hat{P}(X)$ such that each $f^*(\mu)$ is atomless and $\hat{P}(f)\big(f^*(\mu)\big) = \mu$, $\mu \in \hat{P}(Y)$. But if we are interested in atomless maps defined on $Y$, we have the following:

**Theorem 1.3.** *Every open surjection $f\colon X \to Y$ with perfect fibers is a densely atomless Milyutin map provided $X$ and $Y$ are complete spaces.*

Here, a Milyutin map $f\colon X \to Y$ is *densely atomless* if

$$\{g \in Ch_f(Y,X) : g(y) \text{ is atomless for all } y \in Y\}$$

is a dense $G_\delta$-set in the space $Ch_f(Y,X)$ of all choice maps associated with $f$ equipped with the source limitation topology. A few words about this topology. If $(X,d)$ is a bounded (complete) metric space, then there exists a (complete) metric $\hat{d}$ on $\hat{P}(X)$ generating its topology and extending $d$, see [3]. Then $Ch_f(Y,X)$ is a subspace of the function space $C(Y,\hat{P}(X))$ with the source limitation topology whose local base at a given $h \in C(Y,\hat{P}(X))$ consists of all sets

$$B_{\hat{d}}(h,\alpha) = \{g \in C(Y,\hat{P}(X)) : \hat{d}(g(y),h(y)) < \alpha(y) \text{ for all } y \in Y\},$$

where $\alpha$ is a continuous map from $Y$ into $(0,\infty)$. It is well known that this topology does not depend on the metric $\hat{d}$ and it has the Baire property in case $\hat{P}(X)$ is complete. Similarly, $f$ is said to be *densely exact* provided the set

$$\{g \in Ch_f(Y,X) : supp\ g(y) = f^{-1}(y) \text{ for every } y \in Y\}$$

is a dense and $G_\delta$-set in $Ch_f(Y,X)$. When $f$ is both densely atomless and densely exact, it is called densely exact atomless.

**Theorem 1.4.** *Let $f\colon X \to Y$ be an open surjection of complete spaces and $\pi\colon X \to M$ a map into a separable space $M$. Then all choice maps $h \in Ch_f(Y,X)$ such that $\pi(supp\ h(y))$ is dense in $\pi(f^{-1}(y))$ for every $y \in Y$ form a dense $G_\delta$-set in $Ch_f(Y,X)$.*

It is interesting whether in Theorem 1.4 one can substitute the phrase "$\pi(supp\ h(y))$ is dense in $\pi(f^{-1}(y))$" by "$\pi(supp\ h(y) = \pi(f^{-1}(y))$".

Next corollary is a parametrization of the Parthasarathy [12] result that perfect Polish spaces admit atomless measures. It also provides a partial answer of the question [1] whether any open surjection $f$ of complete spaces is an exact atomless Milyutin map provided all fibers of $f$ are perfect Polish spaces.

**Corollary 1.5.** *Let $f\colon X \to Y$ be an open and closed surjection of complete spaces such that all fibers of $f$ are separable (and perfect). Then $f$ is densely exact (atomless) Milyutin map.*

Finally, we generalize [14, Corollary 1.4] and [1, Corollary 1.9] as follows (below a continuous set-valued map means a map which is both lower and upper semi-continuous):

**Corollary 1.6.** *Let $X$ and $Y$ be complete spaces and $\Phi\colon Y \to X$ a continuous set-valued map such that all values $\Phi(y)$ are closed separable subsets of $X$. Then there exists a map $h\colon Y \to \hat{P}(X)$ such that $supp\ h(y) = \Phi(y)$ for every $y \in Y$. If, in addition, all $\Phi(y)$ are perfect sets, the map $h$ can be chosen so that every $h(y)$ is atomless.*

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## 2. Preliminaries

In this section we provide some preliminary results and establish the proof of Theorem 1.1.

Probability Radon measures on a complete space $X$ can be described as positive linear functionals $\mu$ on $C^*(X)$ such that $||\mu|| = 1$ and $\lim \mu(h_\alpha) = 0$ for any decreasing net $\{h_\alpha\} \subset C^*(X)$ which pointwisely converges to 0, see [15]. Under this interpretation, *supp* $\mu$ coincides with the set of all $x \in X$ such that for every neighborhood $U_x$ of $x$ in $X$ there exists $\varphi \in C^*(X)$ such that $\varphi(X\backslash U_x) = 0$ and $\mu(\varphi) \neq 0$. This representation of *supp* $\mu$ easily implies that the set-valued map *supp*: $\hat{P}(X) \to X$ (assigning to each $\mu$ its support) is lower semi-continuous, i.e., $\{\mu \in \hat{P}(X) : supp\ \mu \cap U \neq \varnothing\}$ is open in $\hat{P}(X)$ for any open $U \subset X$. For every closed $F \subset X$, we have $\mu(F) = \inf\{\mu(\varphi) : \varphi \in C(F)\}$ (see for example [7] in case $X$ is compact), where $C(F) = \{\varphi \in C^*(X) : 0 \leq \varphi \leq 1 \text{ and } \varphi(F) = 1\}$.

According to [4], any compatible (complete) metric $d$ on $X$ generates a compatible (complete) metric $\hat{d}$ on $\hat{P}(X)$ such that

$$\hat{d}\big(t\mu + (1-t)\mu', t\nu + (1-t)\nu'\big) \leq t\hat{d}(\mu, \mu') + (1-t)\hat{d}(\nu, \nu')$$

for all $t \in [0, 1]$ and $\mu, \mu', \nu, \nu' \in \hat{P}(X)$. It is easily seen that every ball (open or closed) with respect to $\hat{d}$ is convex.

Let $A_\varepsilon(X)$ denote the set of all $\mu \in \hat{P}(X)$ such that $\mu(\{x\}) \geq \varepsilon$ for some $x \in supp\ \mu$. For any closed $K \subset X$ there exists a closed embedding $i\colon \hat{P}(K) \to \hat{P}(X)$ defined by $i(\nu)(h) = \nu(h|K)$ for all $\nu \in$

$\hat{P}(K)$ and $h \in C^*(X)$. Everywhere below we identify $\hat{P}(K)$ with the set $i\big(\hat{P}(K)\big) = \{\mu \in \hat{P}(X) : supp\ \mu \subset K\}$ which is closed in $\hat{P}(X)$.

**Lemma 2.1.** *Let $X$ be a complete space, $K$ a perfect closed subset of $X$ and $G$ a convex open subset of $\hat{P}(K)$. Then for every $\varepsilon > 0$ we have:*

(1) *$A_\varepsilon(X)$ is a closed subset of $\hat{P}(X)$;*
(2) *$A_\varepsilon(X) \cap \overline{G}$ is a nowhere dense set in the closure $\overline{G}$.*

*Proof.* (1) Since $\hat{P}(X)$ is metrizable, it suffices to check that $\mu_0 = \lim \mu_n \in A_\varepsilon(X)$ for every convergent sequence $\{\mu_n\}_{n\geq 1}$ in $\hat{P}(X)$ with $\{\mu_n\} \subset A_\varepsilon(X)$. To this end, let $H$ be the closure in $X$ of the set $\bigcup_{n\geq 0} supp\ \mu_n$. Because every $\mu \in \hat{P}(X)$ has a separable support, $H$ is a Polish subset of $X$. Considering all $\mu_n$, $n \geq 0$, as elements of $\hat{P}(H)$, we have that the sequence $\{\mu_n\}_{n\geq 1}$ is contained in $A_\varepsilon(H)$ and converges to $\mu_0$. Therefore, by [12, Theorem 8.1], $\mu_0 \in A_\varepsilon(H)$. Consequently, there exists $x_0 \in H$ with $\mu_0(\{x_0\}) \geq \varepsilon$. Therefore, $A_\varepsilon(X)$ is closed in $\hat{P}(X)$.

(2) Since $A_\varepsilon(K) = A_\varepsilon(X) \cap \hat{P}(K)$, it suffices to show that $A_\varepsilon(K)$ is nowhere dense in $\hat{P}(K)$. Suppose $A_\varepsilon(K)$ contains an open subset $W$ of $\hat{P}(K)$ and let $P_\omega(K)$ be the set of all $\mu \in \hat{P}(K)$ having a finite support. Since $P_\omega(K)$ is dense in $\hat{P}(K)$, there exists $\mu_0 = \sum_{i=1}^{i=k} \lambda_i \delta_{x_i} \in P_\omega(K) \cap W$. Here, $\delta_{x_i}$ denotes Dirac's measures at $x_i$ and $\lambda_i = \mu_0(\{x_i\})$. Moreover, $\lambda_i \geq \varepsilon$ for at least one $i$. For each $i \leq k$ and $n \geq 1$ choose a neighborhood $V_i \subset K$ of $x_i$ and $n$ different points $x_{i(1)}, .., x_{i(n)} \in V_i$ such that the family $\{V_i : 1 \leq i \leq k\}$ is disjoint and $dist(x_i, x_{i(j)}) \leq 1/n$ for all $1 \leq j \leq n$. This can be done because $K$ is perfect, so every neighborhood of $x_i$ contains infinitely many points. Consider now the measures $\mu_n = \displaystyle\sum_{i=1}^{i=k}\sum_{j=1}^{j=n} \frac{\lambda_i \delta_{x_{i(j)}}}{n}$. Since $\lim \mu_n = \mu_0$, there exists $n_0$ such that $\mu_n \in W$ for all $n \geq n_0$. Consequently, for every $n \geq n_0$ there exists $i \leq k$ with $\lambda_i/n \geq \varepsilon$, a contradiction. $\square$

**Lemma 2.2.** *Let $f\colon X \to Y$ be an open surjection between complete spaces such that $dim Y = 0$. Then $\hat{P}(f)\colon \hat{P}(X) \to \hat{P}(Y)$ is a soft map.*

*Proof.* According to Theorem 1.3 from [4], it suffices to show that $f$ is everywhere locally invertible. The last notion is defined as follows: for any space $Z$, a point $a \in Z$, a map $g\colon Z \to Y$ and an open set $U \subset X$ with $g(a) \in f(U)$ there exist a neighborhood $V$ of $a$ in $Z$ and a map $h\colon V \to U$ such that $f \circ h = g|V$. Obviously, $f$ is everywhere locally invertible provided it satisfies the following condition:

(*) For any open $U \subset X$ and $a \in f(U)$ there exists a map $g\colon V \to U$ with $V$ being a neighborhood of $a$ in $Y$ such that $f(g(y)) = y$ for all $y \in V$.

To show $f$ satisfies $(*)$, fix an open set $U \subset X$ and $a \in f(U)$. Since $f$ is open, the set $V = f(U) \subset Y$ is also open and the set-valued map $\Phi\colon V \to U$, $\Phi(y) = f^{-1}(y) \cap U$, is lower semi-continuous with closed values. Moreover, $U$ admits a complete metric because $X$ is complete. Then, by the 0-dimensional selection theorem of Michael [11], $\Phi$ has a continuous selection $g$. Obviously, $g$ is as required. □

**Proof of Theorem 1.1.** First, let us show that $\hat{f} = \hat{P}(f)|\hat{P}(f)^{-1}(Y)$ is everywhere locally invertible. It suffices to show that $\hat{f}$ satisfies condition $(*)$ from Lemma 2.2. Suppose that $U \subset \hat{P}(f)^{-1}(Y)$ is open and $y_0 \in \hat{f}(U)$. We need to find a map $\alpha\colon V \to U$, where $V$ is a neighborhood of $y_0$ in $Y$, such that $\hat{f}(\alpha(y)) = y$ for every $y \in V$. To this end, choose a 0-dimensional complete space $Z$ and a perfect Milyutin map $g\colon Z \to Y$, see [6] (recall that a map is perfect if it is closed and has compact fibers). Next, consider the pull-back $T = \{(z,x) \in Z \times X : g(z) = f(x)\}$ of $Z$ and $X$ with respect to the maps $g$ and $f$, and let $p_f\colon T \to Z$, $p_g\colon T \to X$ be the corresponding projections. Since $f$ is open, so is $p_f$. For any $y \in Y$ we have $p_f^{-1}\big(g^{-1}(y)\big) = p_g^{-1}\big(f^{-1}(y)\big) = g^{-1}(y) \times f^{-1}(y)$. Since $g$ is Milyutin, there exists a map $g^*\colon Y \to \hat{P}(Z)$ such that *supp* $g^*(y) \subset g^{-1}(y)$ for all $y \in Y$. Let $\hat{p_f} = \hat{P}(p_f)\colon \hat{P}(T) \to \hat{P}(Z)$ and $\hat{p_g} = \hat{P}(p_g)\colon \hat{P}(T) \to \hat{P}(X)$. Take an open set $G \subset \hat{P}(X)$ with $G \cap \hat{P}(f)^{-1}(Y) = U$ and let $W = \hat{p_g}^{-1}(G)$. Pick $\mu^* \in G \cap \hat{P}(f^{-1}(y_0)$ and let $\nu_0 = \mu_0 \times \mu^*$ be the product measure, where $\mu_0 = g^*(y_0)$. Obviously, $\nu_0 \in \hat{P}(g^{-1}(y_0) \times f^{-1}(y_0)) \subset \hat{P}(T)$. Moreover, $\hat{p_f}(\nu_0) = \mu_0$ and $\nu_0 \in W$ because $\hat{p_g}(\nu_0) = \mu_* \in G$.

Now we can complete the proof that $\hat{f}$ is everywhere locally invertible. Let $g_0\colon \{y_0\} \to \hat{P}(T)$ be the constant map $g_0(y_0) = \nu_0$. Since $\hat{p_f}(\nu_0) = g^*(y_0)$ and, by Lemma 2.2, the map $\hat{p_f}$ is soft, there exists a map $\theta\colon Y \to \hat{P}(T)$ extending $g_0$ such that $\hat{p_f} \circ \theta = g^*$. Obviously, $V = \theta^{-1}(W)$ is a neighborhood of $y_0$, and define $\alpha = \hat{p_g} \circ \theta$. Since for any $y \in V$ we have $\hat{p_f}(\theta(y)) = g^*(y)$, $p_f(\textit{supp}\ \theta(y)) = \textit{supp}\ g^*(y) \subset g^{-1}(y)$ and *supp* $\theta(y) \subset g^{-1}(y) \times f^{-1}(y)$. So, *supp* $\alpha(y) = p_g(\textit{supp}\ \theta(y)) \subset f^{-1}(y)$. Consequently, $\hat{f}(\alpha(y)) = y$. Moreover, $\alpha(y) \in U$ for all $y \in V$.

Since $\hat{f}$ is everywhere locally invertible, by [4, Theorem 1.3], the map $\hat{P}(\hat{f})\colon \hat{P}(\hat{Y}) \to \hat{P}(Y)$ is soft, where $\hat{Y} = \hat{f}^{-1}(Y)$. Moreover, $\hat{P}(X) \subset \hat{P}(\hat{Y}) \subset \hat{P}(\hat{P}(X))$ because $X \subset \hat{Y} \subset \hat{P}(X)$. Therefore the

following diagram

$$\begin{array}{ccc} \hat{P}(\hat{Y}) & \xrightarrow{b_{\hat{P}}} & \hat{P}(X) \\ {\scriptstyle \hat{P}(\hat{f})}\Big\downarrow & & \Big\downarrow{\scriptstyle \hat{P}(f)} \\ \hat{P}(Y) & \xrightarrow{i_{\hat{P}(Y)}} & \hat{P}(Y) \end{array}$$

is commutative. Here, $b_{\hat{P}}$ denotes the restriction $b_{\hat{P}(X)}|\hat{P}(\hat{Y})$ of the barycentric map $b_{\hat{P}(X)}\colon \hat{P}(\hat{P}(X)) \to \hat{P}(X)$, see [3], and $i_{\hat{P}(Y)}$ is the identity on $\hat{P}(Y)$. Since $b_{\hat{P}}$ retracts each $\hat{P}(\hat{f})^{-1}(\mu)$ onto $\hat{P}(f)^{-1}(\mu)$, $\mu \in \hat{P}(Y)$, and $\hat{P}(\hat{f})$ is soft, we finally obtain that $\hat{P}(f)$ is also soft. The proof is completed.

## 3. Atomless Milyutin maps

In this section we provide the proofs of Theorems 1.2 and 1.3.

**Proof of Theorem 1.2**. Suppose that $f\colon X \to Y$ is a surjective atomless Milyutin map with $X$ and $Y$ complete spaces. Then there exists a choice map $h\colon Y \to \hat{P}(X)$ associated with $f$ such that $h(y)$ is an atomless measure for all $y \in Y$. Let $X_0 = \bigcup\{supp\ h(y) : y \in Y\}$ and $f_0 = f|X_0$. Since $f_0^{-1} = supp \circ h$ is lower semi-continuous, $f_0$ is open. Hence, by [1, Theorem 3.6], $X_0$ is complete. Moreover, all $f_0^{-1}(y)$ are perfect sets because $h(y)$ are atomless measures.

For the other implication, assume that $f\colon X \to Y$ is a surjection between complete spaces and there exists a complete subspace $X_0 \subset X$ such that $f_0 = f|X_0$ is an open surjection possessing perfect fibers. Considering $X_0$ and $f_0|X_0$, we may suppose that $f$ is open and all of its fibers $f^{-1}(y)$, $y \in Y$, are perfect sets. Then, by Theorem 1.1, $f$ is Milyutin because $\hat{P}(f)$ has a right inverse as a soft map. To show $f$ is atomless, as in the proof of Theorem 1.1 take a 0-dimensional complete space $Z$ and a perfect Milyutin map $g\colon Z \to Y$. Since $g$ is Milyutin, there exists a map $g^*\colon \hat{P}(Y) \to \hat{P}(Z)$ such that $\hat{P}(g)\big(g^*(\mu)\big) = \mu$ for all $\mu \in \hat{P}(Y)$. By Theorem 1.1, $\hat{P}(f)$ is open (as a soft map). Hence, $\hat{f}\colon \hat{P}(f)^{-1}(Y) \to Y$ is also open (as a restriction of an open map onto a preimage-set). So, the set-valued map $\Phi\colon Z \to \hat{P}(f)^{-1}(Y)$, $\Phi(z) = \hat{f}^{-1}(g(z))$, is lower semi-continuous. Actually, $\Phi(z) = \hat{P}(f^{-1}(g(z)))$ for every $z \in Z$. Let $A_n$, $n \geq 1$, be the set of all $\mu \in \hat{P}(X)$ such that $\mu(\{x\}) \geq 1/n$ for some point $x \in supp\ \mu$. Since the fibers $f^{-1}(y)$ are perfect sets, by Lemma 2.1, $A_n$ are closed in $\hat{P}(X)$ and all intersections $A_n \cap \hat{P}(f^{-1}(y))$ are nowhere dense in $\hat{P}(f^{-1}(y))$, $y \in Y$. Then, by [9, Theorem 1.2], $\Phi$ admits a selection $\theta\colon Z \to \hat{P}(f)^{-1}(Y)$ such that

$\theta(z) \in \Phi(z)\setminus\bigcup_{n=1}^{\infty} A_n$, $z \in Z$. This means that each measure $\theta(z) \in \hat{P}(f^{-1}(g(z)))$ is atomless. The selection $\theta$ generates a regular operator $u\colon C^*(X) \to C^*(Z)$, $u(\phi)(z) = \theta(z)(\phi)$ for all $\phi \in C^*(X)$ and $z \in Z$. Finally, for every $\mu \in P_\beta(Y)$ let $f^*(\mu) \in \hat{P}(X)$ be the measure defined by $f^*(\mu)(\phi) = g^*(\mu)(u(\phi))$, $\phi \in C^*(X)$. It is easily seen that this definition is correct (i.e., $f^*(\mu) \in \hat{P}(X)$) and $f^*\colon P_\beta(Y) \to \hat{P}(X)$ is a continuous map.

Let us show that $\hat{P}(f)\big(f^*(\mu)\big) = \mu$ for every $\mu \in P_\beta(Y)$. It suffices to prove that $f^*(\mu)(\alpha\circ f) = \mu(\alpha)$ for any $\alpha \in C^*(Y)$. And this is really true because $\phi = \alpha \circ f$ is the constant $\alpha(y)$ on each set $f^{-1}(y)$, $y \in Y$. So, $u(\phi)(z) = \theta(z)(\phi) = \alpha(y)$ for any $z \in g^{-1}(y)$. Thus, $u(\phi) = \alpha \circ g$ and $f^*(\mu)(\alpha \circ f) = g^*(\mu)(\alpha \circ g)$. Finally, since $\hat{P}(g)\big(g^*(\mu)\big) = \mu$, we have $g^*(\mu)(\alpha \circ g) = \mu(\alpha)$.

So, it remains to prove only that every $f^*(\mu)$, $\mu \in P_\beta(Y)$, is an atomless measure. To this end, fix $\mu_0 \in P_\beta(Y)$, $x_0 \in supp\ f^*(\mu_0)$ and $\eta > 0$. It suffices to find a function $\phi_0 \in C^*(X)$ with $0 \le \phi_0 \le 1$ such that $\phi_0(x_0) = 1$ and $f^*(\mu_0)(\phi_0) \le \eta$. Since $\theta(z)(\{x_0\}) = 0$, for every $z \in Z$ there exists $\phi_z \in C^*(X)$ and a neighborhood $U_z$ of $z$ in $Z$ such that $0 \le \phi_z \le 1$, $\phi_z(x_0) = 1$ and $\theta(z')(\phi_z) < \eta$ whenever $z' \in U_z$. Using the compactness of $g^{-1}(supp\ \mu_0)$ (recall that $\mu_0$ has a compact support and $g$ is a perfect map), we find neighborhoods $U_{z(i)}$, $i = 1,..,k$, covering $g^{-1}(supp\ \mu_0)$, and let $\phi_0 = \phi_{z(1)} \cdot \phi_{z(2)} \cdot .. \cdot \phi_{z(k)}$. Then $\phi_0$ is as required. Indeed, since $\hat{P}(g)\big(g^*(\mu_0)\big) = \mu_0$, $g^{-1}(supp\ \mu_0)$ contains the support of $g^*(\mu_0)$. Consequently, $g^*(\mu_0)(u(\phi_0)) \le \max\{u(\phi_0)(z) : z \in g^{-1}(supp\ \mu_0)\}$. So, there exists $z_0 \in g^{-1}(supp\ \mu_0)$ such that $g^*(\mu_0)(u(\phi_0)) \le u(\phi_0)(z_0)$. Next, choose $j$ with $z_0 \in U_{z(j)}$ and observe that $\phi_0 \le \phi_j$ implies $u(\phi_0)(z_0) \le u(\phi_j)(z_0) = \theta(z_0)(\phi_j)$. Therefore, $f^*(\mu_0)(\phi_0) \le \theta(z_0)(\phi_j) < \eta$ because $z_0 \in U_{z(j)}$. The proof is completed.

**Proof of Theorem 1.3**. Take a 0-dimensional complete space $Z$, a perfect Milyutin map $g\colon Z \to Y$ and a map $g^*\colon \hat{P}(Y) \to \hat{P}(Z)$ which is a right inverse of $\hat{P}(g)$. We equip $\hat{P}(X)$ with a convex metric $\hat{d}$, and let $A_n$, $n \ge 1$, be the closed subsets of $\hat{P}(X)$ considered in the proof of Theorem 1.2. We need to show that the set $\mathcal{A}$ of all atomless choice maps form a dense $G_\delta$-subset of $Ch_f(Y,X)$. Since each $A_n$ is closed in $\hat{P}(X)$, it is easily seen that the sets

$$\mathcal{U}_n = \{h \in Ch_f(Y,X) : h(y) \notin A_n \text{ for all } y \in Y\}$$

are open in $Ch_f(Y,X)$ and $\mathcal{A} = \bigcap_{n\ge 1} \mathcal{U}_n$. To prove that $\mathcal{A}$ is dense in $Ch_f(Y,X)$, fix $h \in Ch_f(Y,X)$ and a function $\eta\colon Y \to (0,\infty)$. We

are going to find a map $h' \in \mathcal{A}$ such that $\hat{d}(h(y), h'(y)) \leq \eta(y)$ for all $y \in Y$.

Denote by $B(h(g(z)), \eta(g(z)))$ the open ball in $\hat{P}(X)$ (with respect to $\hat{d}$) which is centered at $h(g(z))$ and has a radius $\eta(g(z))$. Define the set-valued map $\Phi\colon Z \to \hat{P}(X)$, $\Phi(z) = \overline{\hat{P}(f^{-1}(g(z))) \cap B(h(g(z)), \eta(g(z)))}$. This is a convex and closed-valued map because any ball in $\hat{P}(X)$ with respect to $\hat{d}$ is convex. Since $\hat{f} = \hat{P}(f)|\big(\hat{P}(f)^{-1}(Y)\big)$ is open (as a soft map, see Theorem 1.1), the set-valued map $z \mapsto \hat{P}(f)^{-1}(g(z))$ is lower semi-continuous. Hence, by [10, Proposition 2.5], so is $\Phi$. Moreover, each $\Phi(z)$ is the closure of the convex open set $\hat{P}(f^{-1}(g(z))) \cap B(h(g(z)), \eta(g(z)))$ in $\hat{P}(f^{-1}(g(z)))$. Hence, according to Lemma 2.1, $A_n \cap \Phi(z)$, $n \geq 1$, are nowhere dense sets in $\Phi(z)$ for every $z \in Z$. Then, by [9, Theorem 1.2], $\Phi$ has a continuous selection $\theta\colon Z \to \hat{P}(X)$ avoiding the set $\bigcup_{n=1}^{\infty} A_n$, i.e., with $\theta(z) \in \Phi(z) \backslash \bigcup_{n=1}^{\infty} A_n$ for every $z \in Z$. Following the notations from the proof of Theorem 1.2, we extend $\theta$ to a map $\bar{\theta}\colon P_\beta(Z) \to \hat{P}(X)$ by $\bar{\theta}(\nu)(\phi) = \nu(u(\phi))$, $\phi \in C^*(X)$. Now let $h'\colon Y \to \hat{P}(X)$ be the composition $\bar{\theta} \circ g^*$. It follows from the proof of Theorem 1.2 that $h'(y)$ is atomless and $h'(y) \in \hat{P}(f^{-1}(y))$ for all $y \in Y$. So, $h' \in \mathcal{A}$.

It remains to show that $\hat{d}(h(y), h'(y)) \leq \eta(y)$, $y \in Y$. To this end, we fix $y \in Y$ and take a sequence $\{\nu_n\} \subset P_\beta(g^{-1}(y))$ converging to $g^*(y)$ such that each $\nu_n$ has a finite support. It is easily seen that if $\nu = \sum_{i=1}^{i=k} t_i \delta_{z(i)} \in P_\beta(g^{-1}(y))$ is a measure with a finite support, then $\bar{\theta}(\nu) = \sum_{i=1}^{i=k} t_i \theta(z(i))$. Since $\hat{d}(\theta(z(i)), h(y)) \leq \eta(y)$ for all $i$ and the metric $\hat{d}$ is convex, we have $\hat{d}(\bar{\theta}(\nu), h(y)) \leq \eta(y)$. In particular, $\hat{d}(\bar{\theta}(\nu_n), h(y)) \leq \eta(y)$ for every $n$. This implies that $\hat{d}(h'(y), h(y)) \leq \eta(y)$ because $h'(y)$ is the limit of the sequence $\{\bar{\theta}(\nu_n)\}$.

## 4. Exact Milyutin maps

In this section the proofs of Theorem 1.4 and Corollaries 1.5-1.6 are established.

**Lemma 4.1.** *Let $U \subset X$ be a non-empty open set in a space $X$. Then the set $\hat{U} = \{\nu \in \hat{P}(X) : supp\ \nu \cap U \neq \varnothing\}$ is open convex and dense in $\hat{P}(X)$.*

*Proof.* Since the support map $\nu \to supp\ \nu$ is a lower semi-continuous map, $\hat{U} \subset \hat{P}(X)$ is open. To show it is dense, suppose there exists an open set $W = \{\nu \in \hat{P}(X) : |\nu(\phi_i) - \nu_0(\phi_i)| < \varepsilon, 1 \leq i \leq k\}$ in $\hat{P}(X)$ with $W \subset \hat{P}(X)\backslash\hat{U}$, where $\phi_i \in C^*(X)$ and $\varepsilon > 0$. We can

suppose that $\nu_0$ has a finite support (recall that the measures with a finite support form a dense set in $\hat{P}(X)$). Let $\nu_0 = \sum_{j=1}^{j=m} \lambda_j \delta_{x(j)}$ such that $\lambda_j > 0$ and $\sum_{j=1}^{j=m} \lambda_j = 1$. Then *supp* $\nu_0 = \{x(j) : 1 \leq j \leq m\} \subset X \backslash U$. Now, let $\nu' = \lambda_0 \delta_{x(0)} + (\lambda_1 - \lambda_0)\delta_{x(1)} + \sum_{j=2}^{j=m} \lambda_j \delta_{x(j)}$, where $x_0 \in U$ and $0 < \lambda_0 < \lambda_1$ such that $\lambda_0 |\phi_i(x_0) - \phi_i(x_1)| < \epsilon$ for every $i = 1, 2, .., k$. The choice of $\lambda_0$ yields that $\nu' \in W$. Consequently, $\nu' \notin \hat{U}$ and *supp* $\nu' \subset X \backslash U$. This contradicts $x_0 \in U \cap$ *supp* $\nu'$.

To show $\hat{U}$ is convex, it suffices to prove that *supp* $\big(t\nu_1 + (1-t)\nu_2\big) =$ *supp* $\nu_1 \cup$ *supp* $\nu_2$ for any $\nu_1, \nu_2 \in \hat{P}(X)$ and any $t \in (0,1)$. Obviously, *supp* $\nu_1 \cup$ *supp* $\nu_2 \supset$ *supp* $\big(t\nu_1 + (1-t)\nu_2\big)$. Assume $x \in$ *supp* $\nu_1$. Then for every neighborhood $V_x$ of $x$ there exists a function $\phi_x \in C^*(X)$ with $\phi_x(X \backslash V_x) = 0$ and $\nu_1(\phi_x) \neq 0$. Since $\nu_1(\phi_x) = \nu_1(\phi_x^+) - \nu_1(\phi_x^-)$, where $\phi_x^+$ and $\phi_x^-$ are the positive and negative parts of $\phi_x$, we can suppose $\phi_x$ is non-negative. Then, $\nu(\phi_x) \geq \nu_1(\phi_x) > 0$ with $\nu = \nu = t\nu_1 + (1-t)\nu_2$. Hence, $x \in$ *supp* $\nu$ which completes the proof. $\square$

**Proof of Theorem 1.4.** Choose a countable base $\{V_n : n \geq 1\}$ for the topology of $M$, and let $B_n = \{\nu \in \hat{P}(X) : supp\ \nu \cap \pi^{-1}(V_n) = \varnothing\}$. By Lemma 4.1, each $B_n$ is closed in $\hat{P}(X)$. Let $\mathcal{B}$ be the set of all maps $h \in Ch_f(Y,X)$ such that $\pi(supp\ h(y))$ is dense in $\pi(f^{-1}(y))$ for any $y \in Y$. Obviously, $\mathcal{B} = \bigcap_{n \geq 1} \mathcal{G}_n$, where $\mathcal{G}_n = \{h \in Ch_f(Y,X) : h(y) \notin B_n$ for all $y \in Y\}$. It suffices to show that each $\mathcal{G}_n$ is open and dense in $Ch_f(Y,X)$ with respect to the source limitation topology.

*Claim* 1. *Each* $\mathcal{G}_n$ *is open in* $Ch_f(Y,X)$.

We can suppose that each $V_n$ is of the form $V_n = g_n^{-1}(0,\infty)$ for some non-negative function $g_n \in C^*(M)$. Then $\nu \in B_n$ if and only if $\nu(g_n \circ \pi) = 0$, $n \geq 1$. Obviously the equality $D_n(\mu,\mu') = \hat{d}(\mu,\mu') + |\mu(g_n \circ \pi) - \mu'(g_n \circ \pi)|$, where $\mu, \mu' \in \hat{P}(X)$ and $\hat{d}$ is a compatible metric on $\hat{P}(X)$, defines a compatible metric on $\hat{P}(X)$ for every $n \geq 1$. Given $h \in \mathcal{G}_n$ we consider the continuous function $\alpha \colon Y \to (0,\infty)$, $\alpha(y) = h(y)(g_n \circ \pi)/2$. We have $B_{D_n}(h,\alpha) \subset \mathcal{G}_n$. Indeed, if $h' \in B_{D_n}(h,\alpha)$, then $|h'(y)(g_n \circ \pi) - h(y)(g_n \circ \pi)| \leq D_n(h(y),h'(y)) < \alpha(y)$ for all $y \in Y$. The last inequality implies $h'(y)(g_n \circ \pi) > \alpha(y) > 0$, $y \in Y$. Hence, $h'(y) \notin B_n$ for all $y \in Y$. So, $h' \in \mathcal{G}_n$ which completes the proof of Claim 1.

To show that any $\mathcal{G}_n$ is dense in $Ch_f(Y,X)$, we fix $m \geq 1$, $h \in Ch_f(Y,X)$ and a function $\eta \colon Y \to (0,\infty)$. We are going to find a map $h' \in \mathcal{G}_m$ with $\hat{d}(h'(y),h(y)) \leq \eta(y)$ for all $y \in Y$. To this end, following the proof of Theorems 1.2 and 1.3, take a complete 0-dimensional space $Z$ and a perfect Milyutin map $g \colon Z \to Y$ with

a right inverse $g^*\colon Y \to P_\beta(Z)$. We also consider the lower semi-continuous convex and closed-valued map $\Phi\colon Z \to \hat{P}(X)$, $\Phi(z) = \hat{P}(f^{-1}(g(z))) \cap B(h(g(z)), \eta(g(z)))$. According to Lemma 4.1, $B_m \cap \hat{P}(f^{-1}(g(z)))$ is a closed nowhere dense subsets of $\hat{P}(f^{-1}(g(z)))$ for every $z \in Z$. Hence, all $B_m \cap \Phi(z)$ are closed and nowhere dense in $\Phi(z)$. Then, by [9, Theorem 1.2], $\Phi$ has a continuous selection $\theta\colon Z \to \hat{P}(X)$ such that $\theta(z) \in \Phi(z)\backslash B_m$, $z \in Z$. As in the proof of Theorem 1.3, let $h'\colon Y \to \hat{P}(X)$ be the composition $\bar{\theta}\circ g^*$, where $\bar{\theta}\colon P_\beta(Z) \to \hat{P}(X)$ is an extension of $\theta$ defined by $\bar{\theta}(\nu)(\phi) = \nu(u(\phi))$, $\phi \in C^*(X)$. Following the arguments from Theorem 1.3, we can show that $\hat{d}(h'(y), h(y)) \leq \eta(y)$ for all $y \in Y$. Next claim completes the proof of Theorem 1.4.

*Claim 2. $h'(y) \notin B_m$ for any $y \in Y$.*

The proof of this claim is reduced to find a function $\phi_y \in C^*(X)$ such that $\phi_y\big(X\backslash\pi^{-1}(V_m)\big) = 0$ and $h(y)(\phi_y) \neq 0$. Indeed, in such a case *supp* $h(y) \cap \pi^{-1}(V_m) \neq \varnothing$. Since $\theta(z) \notin B_m$ for all $z \in g^{-1}(y)$, *supp* $\theta(z) \cap \pi^{-1}(V_m) \neq \varnothing$. Consequently, for any $z \in g^{-1}(y)$ there exists a function $\phi_z \in C^*(X)$ with $\phi_z\big(X\backslash\pi^{-1}(V_m)\big) = 0$ and $\theta(z)(\phi_z) \neq 0$. Considering the positive or negative parts of $\phi_z$, we may assume each $\phi_z \geq 0$. Next, use the continuity of $\theta$ and the compactness of $g^{-1}(y)$ to find finitely many points $z(i) \in g^{-1}(y)$, $i = 1, 2, .., k$, and neighborhoods $U_{z(i)}$ such that $\theta(z)(\phi_{z(i)}) > 0$ provided $z \in U_{z(i)}$. Finally, let $\phi_y = \sum_{i=1}^{i=k} \phi_{z(i)}$. Then $\phi_y\big(X\backslash\pi^{-1}(V_m)\big) = 0$ and $u(\phi_y)(z) = \theta(z)(\phi_y) > 0$ for any $z \in g^{-1}(y)$. So, $h(y)(\phi_y) \geq \min\{u(\phi_y)(z) : z \in g^{-1}(y)\} > 0$ because $g^{-1}(y)$ is compact. This completes the proof of the claim.

**Proof of Corollary 1.5.** Since $f$ is closed with separable fibers, there exists a map $\pi\colon X \to Q$ such that all restrictions $\pi|f^{-1}(y)$, $y \in Y$, are embeddings, see [13]. Here, $Q$ is the Hilbert cube. Then, by Theorem 1.4 (with $M$ replaced by $Q$), $f$ is densely exact. If, in addition, the fibers of $f$ are perfect, both Theorems 1.3 and 1.4 imply that $f$ is densely exact atomless.

**Proof of Corollary 1.6.** Consider the graph $G(\Phi) = \cup\{\{y\} \times \Phi(y) : y \in Y\} \subset Y \times X$ of $\Phi$ and the projection $f\colon G(\Phi) \to Y$. Since $\Phi$ is continuous, $G(\Phi)$ is closed in $Y \times X$ and $f$ is both open and closed. Then $G(\Phi)$ is a complete space. Now, by Corollary 1.5, there exists a map $h'\colon Y \to \hat{P}(G(\Phi))$ with each $h'(y) \in \hat{P}(f^{-1}(y)$ being exact measure. Therefore, *supp* $h'(y) = f^{-1}(y)$. Let $h = \hat{P}(\pi) \circ h'$, where $\pi\colon G(\Phi) \to X$ is the projection into $X$. Since $\pi$ embeds each $f^{-1}(y)$ onto $\Phi(y)$, $h$ is a map from $Y$ into $\hat{P}(X)$ such that *supp* $h(y) = \Phi(y)$ for every $y \in Y$. If $\Phi(y)$ are perfect sets, so are the fibers $f^{-1}(y)$, and

$h'$ can be chosen to be atomless and exact. In such a case $h$ is also atomless.

**Note added in proof.** Recently T. Banakh informed the author that V. Bogachev and A. Kolesnikov [5] proved the following result: The map $\hat{P}(f)$ from Theorem 1.1 is open. This, in combination with Michael's convex-valued selection theorem [10], provides another proof of Theorem 1.1.

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